The concept of infinity

Different meanings through the centuries

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Outline

1. First meanings of infinity
2. Towards an understanding of infinity
3. Counting, mapping and cardinality
4. Cantor‘s paradise and Hilbert‘s hotel
5. Students‘ contributions and observations
6. Conclusion

1. First meanings of infinity

In Contemporary English

- Originates from latin: ad „in-finitum“
- exceedingly great
- having no limit or end, boundless unlimited
- in a mathematical sense: Infinites quantities

2. Towards an understanding of infinity

Examples of the infinity concept in classrooms:

- Number of numbers, number of prime numbers
- Geometric lines
- Asymptotic behaviour of functions, e.g. f(x) = 1/x
- The symbol ∞ ($n \to \infty$, $x \to a$)

Of the different "kinds" of infinity mainly remains:

- Limit conception, due to Weierstrass

3. Counting, mapping and cardinality

Georg Cantor (1845 - 1918)

- Elementary definition of a set, and also:
- 1-1-mappings between two sets to identify they common

  cardinality

  - Relational aspect
  - Functional aspect
  - One consistant view including finite and infinite sets

3. Counting, mapping and cardinality

Beiträge zur Begründung der transfiniten Mengenlehre.

Von

Georg Cantor in Halle a./S.

(Erster Artikel.)

„Hypotheses non fingo.“

„Neque enim leges intellectui aut rebus damus ad arbitrium nostrum, sed tanquam scribae fideles ab ipsius naturae voce latas et prolatas excipimus et describimus.“

„Veniet tempus, quo ista quae nunc latent, in lucem dies extrahat et longioris aevi diligentia.“

§ 1.

Der Mächtigkeitsbegriff oder die Cardinalzahl.

Unter einer ‚Menge‘ verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objecten m unsrer Anschauung oder unseres Denkens (welche die ‚Elemente‘ von M genannt werden) zu einem Ganzen.

In Zeichen drücken wir dies so aus:

(1) $$M = \{m\}.$$

Die Vereinigung mehrerer Mengen $M, N, P, \ldots$, die keine gemeinsamen Elemente haben, zu einer einzigen bezeichnen wir mit

(2) $$(M, N, P, \ldots).$$

Die Elemente dieser Menge sind also die Elemente von M, von N, von P etc. zusammengenommen.

‚Theil‘ oder ‚Theilmenge‘ einer Menge M nennen wir jede *andere* Menge M_1, deren Elemente zugleich Elemente von M sind.

3. Counting, mapping and cardinality

[481] CONTRIBUTIONS TO THE FOUNDING OF THE THEORY OF TRANSFINITE NUMBERS

(First Article)

> "Hypotheses non fingo."
>
> "Neque enim leges intellectui aut rebus damus ad arbitrium nostrum, sed tanquam scribæ fideles ab ipsius naturæ voce latas et prolatas excipimus et describimus."
>
> "Veniet tempus, quo ista quæ nunc latent, in lucem dies extrahat et longioris ævi diligentia."

§ 1

The Conception of Power or Cardinal Number

By an "aggregate" (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects *m* of our intuition or our thought. These objects are called the "elements" of M.

In signs we express this thus:

(1) $$M = \{m\}.$$

We denote the uniting of many aggregates M, N, P, . . ., which have no common elements, into a single aggregate by

(2) $$(M, N, P, \ldots).$$

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3. Counting, mapping and cardinality

Definition of Aleph

$\aleph_0$ is the cardinality of the set of the natural numbers, and is the “smallest” infinite cardinal. A set has cardinality $\aleph_0$, if there exists a 1-1-mapping between this set and the natural numbers. Such sets are called *countable*.

Examples are:

- (of course) the set of the natural numbers
- the set of all integers
- the set of all prime numbers
- the set of all rational numbers

Counterexamples:

- The set of the real numbers is an uncountable sets, i.e. there is no bijection between and . Its cardinality is denoted by $\aleph_1$.
- If c denotes the cardinality of and $\aleph_1$ is the cardinal number following $\aleph_0$, the continuum hypothesis states c = $\aleph_1$.

Arithmetic of cardinal numbers: First examples

$$1 + \aleph_0 = \aleph_0,$$

$$n + \aleph_0 = \aleph_0$$

for any $n \in \mathbb{N}$,

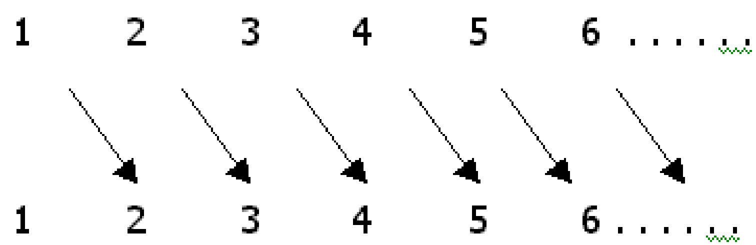
as well as

$$2\aleph_0 = \aleph_0.$$

4. Cantor‘s[1]) paradise and Hilbert‘s[2]) hotel

Hilbert:

- “From the paradise, that Cantor created for us, no-one can expel us.“
- Presentation of one standard version of the well-known Hotel Hilbert.



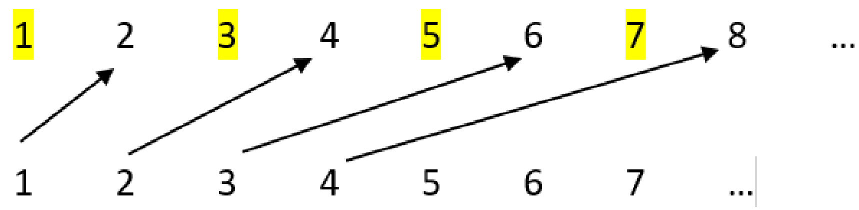
[1]) 1845 – 1918
[2]) 1862 – 1943

Contributions in detail

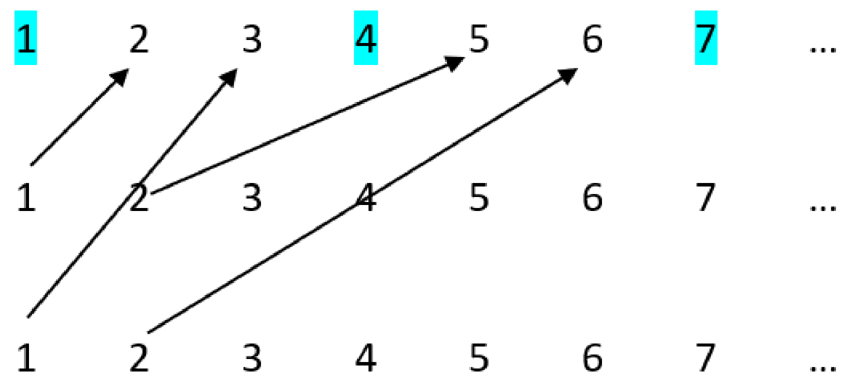
- Well-known ones (re-inventions) and others
- Different starting points creating new examples and

  their solutions

5. Students' contributions and observations during seminars



Filling the gaps after doubling. This applies for any finite number of „Hilbertian busses".

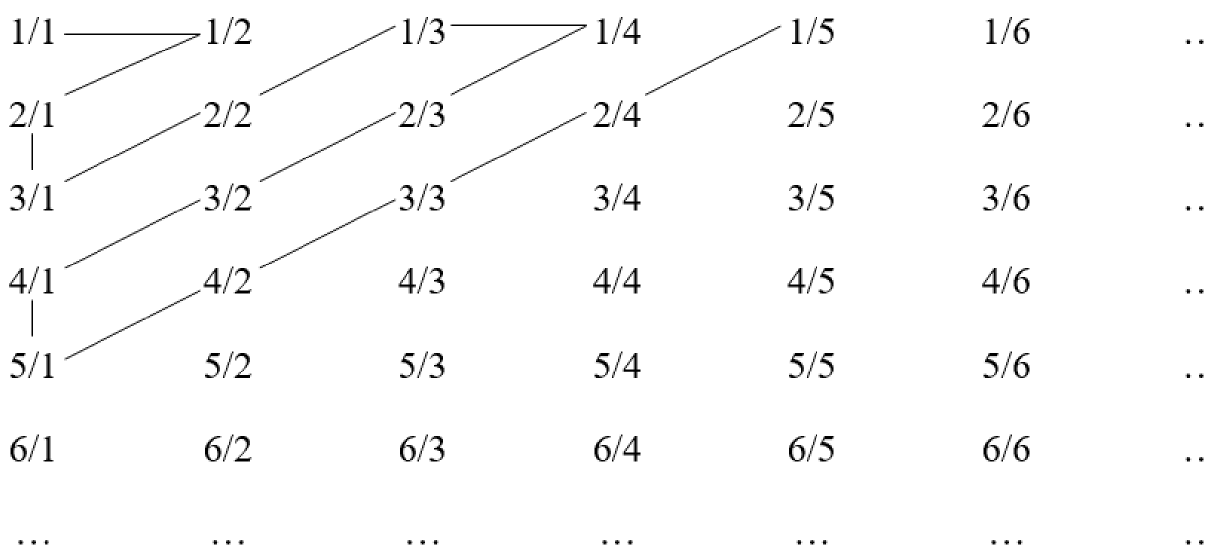


Variation: Filling wider gaps (after multiplying with 3) in the case oft two „Hilbertian busses".

Examples of solutions

- Doubling previous guest's room numbers
- Mulitiplication previous guest's room numbers in a proper way
- Alternating or accordingly periodic entrance of the new guests
- Alternative patterns equivalent to the diagonal method

Examples: Cantor's first diagonal argument (ist countable)



5. Students' contributions and observations during seminars

Examples: Cantor's second diagonal argument (is not countable)

$$\alpha_1 = 0,\underline{\alpha_{11}}\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}.....$$

$$\alpha_2 = 0,\alpha_{21}\underline{\alpha_{22}}\alpha_{23}\alpha_{24}\alpha_{25}.....$$

$$\alpha_3 = 0,\alpha_{31}\alpha_{32}\underline{\alpha_{33}}\alpha_{34}\alpha_{35}.....$$

$$\alpha_4 = 0,\alpha_{41}\alpha_{42}\alpha_{43}\underline{\alpha_{44}}\alpha_{45}.....$$

$$.....$$

$$\beta = 0,\beta_1\beta_2\beta_3... \in [0,\ 1[\text{ such that}$$

$$\beta_i = \begin{cases} 1, & if\ \alpha_{ii} \neq 1 \\ 7, & if\ \alpha_{ii} = 1 \end{cases}$$

6. Conclusions

- Student‘s experience with an (partly) accessible original text
- Arithmetical transfer
- Concept of function and bijectivity
- Learning something about mathematical genesis, conceptualisation (and even controversy)
- Empowering future teachers to conduct mathematical discussions in classroom

References

Cantor Georg (1895): Beiträge zur Begründung der transfiniten Mengenlehre. In: Mathematische Annalen *46*, 481–512.

Dieudonné, Jean (1985): Geschichte der Mathematik 1700-1900. Braunschweig, Wiesbaden: Vieweg.

Dötschel, Deborah (2011): Zum Verständnis der Unendlichkeitsbegriffs im Mathematikunterricht. In: Beiträge zum Mathematikunterricht *45*, 207 – 210.

Euclides (1997): Die Elemente: Bücher I – XIII. Translated by Clemens Thaer, 3. Aufl.. Leipzig: Akad. Verl-Ges. Geest und Portig, 1922–.

Freudenthal, Hans (1983): Didactical Phenomenology of Mathematical Structures. Dordrecht: D. Reidel Publishing Company.

Kattou Maria; Michael Thanasia; Kontoyianni Katerina; Christou Constantinos; Philippou George (2009): Teachers' perceptions about infinity: a process or an object? In: Proceedings of CERME 6, p. 1771 – 1780.

Rudy Rucker (1982): Infinity and the Mind. Brighton: John Spiers, Brighton.

Vollrath, Hans-Joachim (1987): Begriffsbildung als schöpferisches Tun im Mathematikunterricht. Zentralblatt für Didaktik der Mathematik *19*, 123-127.

Winter, Heinrich (1996): Mathematikunterricht und Allgemeinbildung. In: Mitteilungen der Gesellschaft für Didaktik der Mathematik, *61*, 37–46.

Woerner, Deborah (2013): Zum Verständnis des Unendlichkeitsbegriff im Mathematikunterricht – eine empirische Untersuchung. In: Beiträge zum Mathematikunterricht *47*, 1110 – 1113.

References

German:
http://gdz.sub.uni-goettingen.de/dms/load/img/?PID=GDZPPN00225557X

English:
https://archive.org/details/contributionstot003626mbp

Thank you for your attention!

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